Gonality of Complete intersection curves

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Let C be a projective curve, $\tilde{C} \to C$ its normalization.
Then the gonality of C is the minimum degree of a map $\tilde{C} \twoheadrightarrow \mathbb{P}^1$ i.e. $\mathrm{gon}(C) = \min\{\deg L \mid L$ b.p.f. l.b on $\tilde{C}$ w/ $h^0(L) \geq 2\}$.

(What about projection from a point?)

Ex: Let C be a smooth plane curve of deg $d \geq 2$. Then projecting from a point on C gives $\mathrm{gon}(C) \leq d-1$.

Exercise: Show $\mathrm{gon}(C) = d-1$.

More generally...

Thm: $C \subset \mathbb{P}^r$ a smooth complete intersection of hypersurfaces of degrees $2 \leq a_1 \leq a_2 \leq a_{r-1}$.
Then $\mathrm{gon}(C) \geq (a_1 - 1) \cdot a_2 \cdot \ldots \cdot a_{r-1}$.
Moreover, there exists a C where equality holds.

Pf: Set $\gamma = a_3 a_4 \cdot \ldots \cdot a_{r-1}$.

Let $X \supset C$ be general complete intersection 3-fold of type $(a_3, \ldots, a_{r-1})$. If $r=3$, let $X = \mathbb{P}^3$, $\gamma = 1$.

Let $f: Y \longrightarrow X$ be the blowup along C,

$E \subset Y$ the exceptional divisor, $\pi : E \to C$ the natural map.

Define $\mathcal{A} = \pi^* A$. Note that $\mathcal{A}$ is also globally generated, and since it is pulled back from C, we can find a base-point free pencil of sections (pullback preserves surjections)

$$\mathcal{O}_Y^2 \twoheadrightarrow \mathcal{A} \qquad \text{Let } \mathcal{A} = \mathcal{O}(B_A),\ B_A \text{ eff.}$$

Get SES on Y:

$$0 \to F \to \mathcal{O}_Y^2 \to \mathcal{A} \to 0$$

F is a vector bundle of rk 2 (same argument as for surfaces — Take local Tor on stalks)

$\boxed{\text{Step 1}}$

Chern classes of F?

$$\det(F) \cong \mathcal{O}(-E) \Rightarrow c_1(F) = -E$$

Need to calculate $c_2(\mathcal{A})$...

<u>Exercise</u>: If $Z \subseteq Y$ is a subsch. of codim n, and $\mathcal{F}$ a sheaf w/ support $= Z$ then $c_n(\mathcal{F}) = (-1)^{n+1}[Z]$

Then $0 \to \mathcal{O}_Y(-E) \to \mathcal{O}_Y \to \mathcal{O}_E \to 0$

$$\Rightarrow \quad c_2(\mathcal{O}_E) = -\underset{\overset{\parallel}{0}}{c_2(\mathcal{O}_Y(-E))} - \underset{\overset{\parallel}{-E}}{c_1(\mathcal{O}_Y(-E))} \cdot \underset{\overset{\parallel}{E}}{c_1(\mathcal{O}_E)} = E^2$$

Pushing forward ideal sequence:

$$0 \longrightarrow \mathcal{O}_E \rightarrow \mathcal{A} \longrightarrow \mathcal{O}_{B_A}(\mathcal{A}) \rightarrow 0$$

$$\Rightarrow c_t(\mathcal{A}) = (1 + Et + E^2t^2 + \ldots)(1 - B_A t^2 + \ldots)$$

$$= 1 + Et + (E^2 - B_A)t^2 + \ldots$$

$$\Rightarrow c_2(\mathcal{A}) = E^2 - B_A$$

So returning to $0 \rightarrow F \rightarrow \mathcal{O}_Y^2 \longrightarrow \mathcal{A} \rightarrow 0$

$$c_2(F) = -\underset{\overset{\parallel}{E}}{c_1(\mathcal{A})} \cdot \underset{\overset{\parallel}{-E}}{c_1(F)} - c_2(\mathcal{A}) = E^2 - E^2 + B_A = B_A$$

$\boxed{\text{Step 2}}$ Apply generalized Bogomolov instability to F:

Thm: (Miyaoka) F a rank 2 v.b. on a sm. 3-fold. Fix D globally generated and D' ample.

If $(c_1(F)^2 - 4c_2(F)) \cdot D' > 0$, l

then there's a rank one subsheaf $L \subset F$ s.t.

$$(2c_1(L) - c_1(F)) \cdot D' \cdot D > 0.$$

Let H = pullback to Y of hyperplane section on X.

For $0 \leq \varepsilon \in \mathbb{Q}$, define $D_\varepsilon = (a_2 + \varepsilon)H - E$.

Set $D = D_0$. Strategy, use D, D_ε in Miyaoka's Thm.

Notice that $|D| \cong |V|$, where $V \subseteq H^0(\mathcal{O}_X(a_2))$ is the locus containing C.

Exercise: Show $|V|$ is b.p.f. away from C and separates normal vectors along C.

Exercise $\Rightarrow$ D is globally generated.

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Claim: D_ε is ample for $\varepsilon > 0$.

Pf: Nakai-Moishezon $\Rightarrow$ Need to check $D_\varepsilon^{\dim(V)} \cdot V > 0$ $\forall$ V.

Check for curves: Can take an effective representative $S \in |D|$ so that S is the strict transform of a c.I. surface of type $(a_2, \ldots, a_{r-1})$ containing C, and $D_\varepsilon = S + \varepsilon H$

Take $C' \subset Y$ an irreducible curve.

Case 1: $C' \not\subset E$. Then choose a hyperplane section on X that meets the image of C' in X away from E.

Then $\varepsilon H . C' > 0$, and $|D|$ is globally generated, so we can choose $S \not\supset C' \Rightarrow S . C' \geq 0$.

Case 2: $C' \subset E$. Again, we can choose S not containing C' by global generation.

If $S \cdot C' = 0$, then C' must intersect every fiber of E (as a ruled surface). Choose a hyperplane section on X intersecting C transversely. Then the pullback will be a union of fibers $\Rightarrow \varepsilon H > 0$.

Exercise: Show $D_\varepsilon^3 > 0$ and $D_\varepsilon^2 \cdot V > 0 \ \forall$ surfaces $V \subset Y$.
Hint: Lefschetz Hyperplane $\Rightarrow \mathrm{Pic}(Y) = \mathbb{Z} \oplus \mathbb{Z}$.

Now, we need to show $(*) > 0$, where $(*) =$

$$\left(c_1(F)^2 - 4c_2(F)\right)\left(D_\varepsilon\right) = \left((-E)^2 - 4B_A\right)\underbrace{\left((a_2+\varepsilon)H - E\right)}_{= \varepsilon H + S}$$

$$= \varepsilon E^2 H + E^2 S - 4\varepsilon B_A H - 4 B_A S$$

We can choose H to avoid finitely many fibers of E, so $B_A H = 0$

S intersects E in a section, so
$B_A S = \deg A = d$ (*draw*)

$E^2 H$ calculation:

C intersects a gen. hyperplane section H' in S in $a_1 a_2 \gamma$ points, so

$$EH = \{a_1 a_2 \gamma \text{ fibers}\}$$

$E = (a_2 H - S)$. We can choose H to avoid finitely many fibers in E, and S intersects E in a section.

Thus $E^2 H = 0 - a_1 a_2 \gamma = \boxed{-a_1 a_2 \gamma}$

$E^2 S$ calculation:

$$E^2 S = E^2 (a_2 H - E) = -a_1 a_2^2 \gamma - E^3$$

Exercise: $E^3 = -\deg N_{C/X}$.

Have normal bundle SES:

$$0 \to N_{C/X} \to N_{C/\mathbb{P}^r} \to N_{X/\mathbb{P}^r}\big|_C \to 0$$

$$0 \to \bigoplus_{i=3}^{r-1} \mathcal{O}_C(a_i) \to \bigoplus_{i=1}^{r-1} \mathcal{O}_C(a_i) \to N_{C/X} \to 0$$

So $\det N_{C/X} = \mathcal{O}_C(a_1 + a_2)$

$$\Rightarrow \deg N_{C/X} = (a_1 + a_2)\, a_1 a_2 \gamma$$

$$\Rightarrow E^3 = -a_1^2 a_2 \gamma - a_1 a_2^2 \gamma$$

So $E^2 S = -a_1 a_2^2 \gamma + a_1^2 a_2 \gamma + a_1 a_2^2 \gamma = a_1^2 a_2 \gamma$

Thus $(*) = -\varepsilon a_1 a_2 \gamma + a_1^2 a_2 \gamma - 4d = (a_1 - \varepsilon)\, a_1 a_2 \gamma - 4d.$

Assume for the sake of contradiction that $d < (a_1 - 1)\, a_2 \gamma$.

Then $\left(\frac{a_1^2}{a_1 - 1}\right) d < a_1^2 a_2 \gamma$ (recall $a_1 \geq 2$)

If $a_1 = 2$, $\frac{a_1^2}{a_1 - 1} = 4$.

For $a_1 > 2$, $\frac{a_1^2}{a_1 - 1}$ is increasing, so

$$4d \leq \left(\frac{a_1^2}{a_1 - 1}\right) d < a_1^2 a_2 \gamma.$$

Thus for sufficiently small $\varepsilon > 0$, $4d < (a_1 - \varepsilon)\, a_1 a_2 \gamma$

$\Rightarrow (*) > 0$. Fix ε so that this holds.

So Miyaoka's Theorem applies, and we get a rank one subsheaf $L \subset F$ such that

$(2c_1(L) - c_1(F))D_\varepsilon \cdot D > 0$

$\boxed{\text{Step 3:}}$ Understand L.

<u>Note</u>: We can assume L is a line bundle.

Otherwise, $L^{**} \subset F^{**} = F$

<u>Fact</u>: Reflexive sheaves of rk 1 on smooth varieties are invertible.

Since $L \to L^{**}$ is an isomorphism away from a codim 2 locus, $c_1(L) = c_1(L^{**})$, so we can replace L with L^{**} (reflexive hull).

$\boxed{\text{Feb 22}}$

<u>Claim</u>: We can assume $L \to F$ drops rank (if at all) on a codimension 2 subset $Z \subset Y$.

<u>Pf of claim</u>: First of all, since $\mathrm{rk}\, F = 2$, codim Z <u>can</u>'t be 3.

Assume $L \to F$ drops rank along an effective divisor B (counting multiplicity).

Then it factors through
$$\begin{tikzcd} L \arrow[rr] \arrow[dr] & & F \\ & L(B) \arrow[ur] & \end{tikzcd}$$

$L(B) \to F$ doesn't drop rank along a divisor,
and $c_1(L(B)) = c_1(L) + B$, so it will only intersect $\underset{\substack{\uparrow \\ \text{ample}}}{D_\varepsilon} \cdot \underset{\substack{\uparrow \\ \text{nef}}}{D}$
more positively.

Thus, we can safely replace L with $L(B)$. $\square$

Recall that $\mathrm{Pic}(X) = \mathbb{Z}$, so

$$L = \mathcal{O}_Y(-tH - \mu E) \text{ for some } t, \mu \in \mathbb{Z}.$$

$\boxed{\text{Step 4:}}$ Restrict to surface + get a contradiction.

Set $F' = F|_S$

$$L|_S = \mathcal{O}_S(-tH - \mu E) = \mathcal{O}_S(-tH - \mu \underbrace{C}_{= a_1 H}) = \mathcal{O}_S(-(t + \mu a_1)H)$$

Set $\alpha = t + \mu a_1$.

Since $L \hookrightarrow F$ drops rank (if at all) in codim 2, restriction to S will preserve the inclusion, and we get

$$0 \to \mathcal{O}_S(-\alpha H) \to F' \to M \otimes I_W \to 0$$

where M is a line bundle and $W \subset S$ is finite.

$$c_1(F') = c_1(F)|_S = [-E]|_S = -C = -a_1 H$$

$$\Rightarrow c_1(M) = -a_1 H + \alpha H \Rightarrow M = \mathcal{O}_S\big((\alpha - a_1)H\big)$$

So SES becomes

$$0 \to \mathcal{O}_S(-\alpha H) \to F' \to \mathcal{O}_S\big((\alpha - a_1)H\big) \otimes I_W \to 0$$

By instability,

$(2c_1(L) - c_1(F)) \cdot D_\varepsilon \cdot D > 0$, where $D = S$ (up to linear equivalence)

$$\Rightarrow \big(-2\alpha H - (-a_1 H)\big) \cdot \overbrace{\big((a_2+\varepsilon)H - \underbrace{C}_{a_1 H}\big)}^{D_\varepsilon|_S} > 0$$

$$(a_1 - 2\alpha)\underbrace{(a_2 - a_1 + \varepsilon)}_{>0}\underbrace{H^2}_{>0} > 0$$

So $\boxed{a_1 > 2\alpha}$ (i)

Now we can calculate $c_2(F)$ in two ways:

$$c_2(F') = \underbrace{c_2(F)}_{[B_A]}\big|_S = d.$$

And, from a previous calculation,

$$c_2(F) = (-\alpha H)\big((\alpha - a_1)H\big) + \text{length}(W)$$

$$H \cdot H = \deg S \text{ in } \mathbb{P}^{r-1} = a_2 \gamma$$

So $c_2(F) = (a_1\alpha - \alpha^2)\, a_2 \gamma + \text{length}(W)$

$$\Longrightarrow \left(a_1\alpha - \alpha^2\right) a_2 \gamma \le d < (a_1 - 1) a_2 \gamma$$

$$\Longrightarrow a_1\alpha - \alpha^2 < a_1 - 1$$

$$\Longrightarrow (\alpha - 1) a_1 < \alpha^2 - 1 = (\alpha + 1)(\alpha - 1) \quad \text{(ii.)}$$

<u>Claim</u>: $h^0(F') = 0$.

By the claim, $\alpha > 0$. And (ii) $\Rightarrow \alpha \neq 1$.

Thus, $a_1 < \alpha + 1$. By (i) and the fact that $a_1 \ge 2$, we get a contradiction.

<u>Pf of claim</u>: Recall that on Y,

$$0 \to F \to \mathcal{O}_Y^2 \to \mathcal{A} \to 0,$$

where, by construction, $\mathcal{O}_Y^2 \to \mathcal{A}$ is injective on global sections.
so $h^0(F) = 0$

$F \to \mathcal{O}_Y^2$ drops rank along E, which doesn't contain S.

Thus, the restriction $F' \to \mathcal{O}_X^2$ is still injective, and we get

$0 \to F' \to \mathcal{O}_X^2 \to A \to 0$ where $\mathcal{O}_X^2 \to A$ is still injective on global sections.

Thus $h^0(F') = 0$. $\square$

<u>Exercise</u>: Show that the inequality in the Theorem is strict by

finding such a C with $\text{gon}(C) = (a_1 - 1) a_2 \cdot \ldots \cdot a_{r-1}$.

Open questions:

1.) This theorem gives a lower bound for $\text{gon}(C)$. We can expect the gonality to be higher in general. What is $\text{gon}(C)$ for a (very) general complete intersection?

2.) Let S be a smooth surface. Define $\text{irr}(S)$ = degree of irrationality of S = $\min\{\deg f \mid f : S \dashrightarrow \mathbb{P}^2 \text{ is dominant}\}$.

Can we use a similar method to give a lower bound for $\text{irr}(S)$ where S is a complete intersection?

Idea for 2: If A on S has reduced base points $P_1, \ldots, P_n$, blowup S at $P_1, \ldots, P_n$

$$\begin{array}{ccc} \mathcal{A} = \pi^* A & \tilde{S} \subseteq Y & \\ & \downarrow \pi \quad \downarrow & \leftarrow \text{blowup at } P_1, \ldots, P_n \\ A & S \subseteq X & \end{array}$$

Then $\mathcal{A}(-E)$ is globally generated....